Finite element approximation of Helmholtz problems for seismic wave propagation

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Abstract

In d dimensions, achieving a precise approximation of a varied function oscillating at frequency k demands approximately k^d degrees of freedom. When addressing the Helmholtz equation in a numerical manner (with a wavenumber k within d dimensions), encountering the pollution effect arises when, as k approaches infinity, the total necessary degrees of freedom for sustaining accuracy escalates at a rate surpassing this inherent threshold. For domain-centered approaches like finite element methods, this threshold grows faster than k^d. The development of numerical methods to solve the Helmholtz equation, which behave robustly concerning the wave number, is a topic of intense research.

Keywords: Helmholtz Equation , High frequency ,Pollution Effect ,generalized FEM, MMAm

1 Introduction

Seismic waves provide valuable insights into Earth's structure and seismic events. Advanced numerical methods play a pivotal role in accurately modeling these waves, handling vast datasets, and improving our understanding and predictive capabilities in seismology, earthquake studies, and resource exploration.

Such waves are governed by the wave equation :

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = g$$

In our case, the inhomogeneity term g is time dependant :

$$g(x,t) = f(x)exp^{ikt}$$

The solution of the wave equation is of the form $w(x,t) = u(x)exp^{ikt}$, where the amplitude $u(x)$ satisfies the Helmholtz equation :

$$-\Delta u - k^2 u = f \qquad (\star)$$

Due to the central role of the Helmholtz equation in linear wave propagation, significant attention has been dedicated to examining the characteristics of its solutions, e.g their asymptotic behaviour as k approaches infinity, and devising efficient methods for their computational determination.

In one-dimensional homogeneous media, Babuska and Ihlenburg , in their papers Ihlenburg and Babuška (1997), have established optimal pre-asymptotic error estimates for Lagrangian polynomial discretizations. The error is dissected into two distinct components: the best approximation error, characterized by the order $w^p h^p$, and the phase lag, with an order of $w^{2p+1hh}2p$.

Notably, the error remains bounded regardless of frequency when the mesh step is approximately $h = w^{-1-1/(2p)}$ The pollution effect persists for all p, as the mesh step must satisfy $h = w^{-1-1/(2p)} << w^{-1}$. Similarly, in the heterogeneous case, Frelet (2015) analyzed traditional high-order discretization methods that frequently struggle to accommodate propagation media with fine-scale variations in the

velocity parameter. This is because these methods typically utilize coarser meshes compared to low-order techniques, making it challenging to capture the intricate details of the velocity field. If the velocity parameter is enforced to be constant within each cell, either through averaging or local homogenization strategies, there is a risk of losing fine-scale information, albeit partially. Additionally, constraining the mesh step to maintain a constant velocity parameter inside each cell often results in a significant increase in computational costs, especially in cases where the medium exhibits high heterogeneity.

In this paper, we will focus on the Helmholtz equation in both a homogeneous and a heterogeneous medium, providing results regarding stability by going over resullts that were previously stated and proven.

1.1 What is the pollution effect ?

When using numerical methods like finite element methods (FEM) or finite difference methods (FDM) to solve the Helmholtz equation, the discretization of space leads to a system of algebraic equations.

The issue of pollution effect emerges as k increases. It's observed that the number of elements or degrees of freedom needed to maintain accuracy grows rapidly with k. Mathematically, the pollution effect manifests as the computational requirements increasing faster than the natural expectation for maintaining accuracy.

2 The homogeneous Helmholtz equation

2.1 Finite element spaces and Galerkin discretization

The Dirichlet-to-Neumann transformation:

The straightforward application of finite element or finite difference methods assumes discretization over a bounded or finite domain. However, the Helmholtz equation is defined on a unbounded domain. To resolve this issue, We introduce a ball B such that $D \subset\subset B$, thereby enabling the transformation of exterior conditions into non-local conditions on ∂B.

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Thus, the Helmholtz equation must be solved on the finite domain: $\Omega = B \cap R^d \setminus \bar{D}$

The equation (*) becomes:

$$\begin{cases} \text{-}\Delta u - k^2 u = f \text{ on } \Omega \\ \text{u} = 0 \quad on \ \ \partial D \\ \frac{\partial u}{\partial n} - iku = q \text{ sur } \partial B \end{cases} \tag{1}$$

2.1.1 The variational formulation

Transformation of the problem: the variational formulation

Let $H^1(\Omega)$ be the Sobolev space defined on Ω. Thus, we introduce: $V = \{v \in H^1 : v_{|\partial D} = 0\}$.

We seek $u \in H^1(\Omega)$ such that: $a(u, v) = F(v)$ for all $v \in V$ (1.2).

where: $a(u, v) = \int_\Omega \nabla u \nabla v - k^2 uv - ik \int_{\partial B} uv$ and $F(v) = \int_\Omega fv + \int_{\partial B} qv$

Discretization Elements:

The Galerkin finite element method is obtained by replacing in equation (1) the infinite-dimensional space V with spaces of finite elements.

Let $\tau = \{\delta_1, \delta_2, ..., \delta_n\}$ denote a mesh of finite elements composed of simplicial or quadrilateral elements. The mesh size is denoted by $h = \max_{\delta \in \tau} \text{diam } \delta$

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Let $\Theta = \{x_1, x_2, ..., x_n\}$ denote the set of vertices of τ that are not on the essential boundary ∂D. For x_i belonging to Ω, the local basis functions are given by:

$$\begin{cases} \phi(x_i) = \delta_{i,j}, \quad 1 \leq i, j \leq n \\ \phi|_\delta \text{ is bilinear for every } \delta \in \tau \end{cases}$$

The space of finite elements corresponding to the grid T is given by:

$$V_h = \text{Vect}\{\phi_i \mid 1 \leq i \leq n\}$$

Therefore, the estimated solution u_h is: $u_h = \sum_{i=0}^{n} U_i \phi_i$ where U_i are scalars to be determined. (1) becomes:

$$G^{Gal} u_h = Q^{Gal}(f, q) \qquad (1^*)$$

with:

$$G^{Gal}_{i,j} = a(\phi_j, \phi_i),\ 1 \leq i, j \leq n \text{ and } Q^{Gal}(f, q) = \{F(\phi_i)\}_{1 \leq i \leq n}$$

2.2 On the stabilization of the Helmholtz equation in one dimension

Here, we demonstrate that in one dimension, there exists a Generalized Finite Element Method **GFEM** without pollution for the Helmholtz problem (Ihlenburg and Babuška (1997)).

Without loss of generality, we consider the following problem:

$$\begin{cases} -u'' - k^2 u = f \qquad \Omega = (0, 1) \\ u(0) = 0 \\ u'(1) - iku(1) = 0 \end{cases} \tag{2}$$

$\{x_i\}$ denotes a set of grid points, and the grid τ_h is composed of intervals $\Delta_i = [x_{i-1}, x_i]$. The discretization of (5) yields a system of difference equations :

$$G^{Gal}_{i-1,i} u_{i-1} + G^{Gal}_{i,i} u_i + G^{Gal}_{i,i+1} u_{i+1} = \int_{\Delta_i} f\phi_i + \int_{\Delta_{i+1}} f\phi_i$$

B Showed that G and Q can be chosen in the following manner :

$$G^{stab}_{i,j} = \frac{k^2 h}{2tan(\frac{kh}{2})} \begin{cases} \dfrac{sin(k(x_{i+1} - x_{i-1}))}{sin(k(x_{i+1} - x_i))sin(k(x_i - x_{i-1}))} & si \ \ i = j \leq n \\ \dfrac{e^{-ik(x_n - x_{n-1})}}{sin(k(x_n - x_{n-1}))} & si \ \ i = j = n \\ \dfrac{-1}{sin(k(x_i - x_j))} & si \ \ |i - j| = 1 \\ 0 & otherwise \end{cases}$$

$$\text{and } (\mathrm{Q}_{stab} f)_i = \frac{h}{2tan(\frac{kh}{2})} \sum_{i}^{min(i+1,n)} \frac{tan(k\frac{x_m - x_{m-1}}{2})}{x_m - x_{m-1}} \frac{\int_{x_{m-1}}^{x_m} f(x)dx}{x_m - x_{m-1}}$$

Definition 2.1. *Let W be a subspace of $V' \times H^{-1/2}(\partial B)$. For $(f, q) \in W$, let $U_{f,q}$ be the solution of the Helmholtz problem. We say that a finite element method exhibits the pollution effect if there exist numbers r, s belonging to R, and $t > 0$ such that the error of the best approximation satisfies:*

$$\|e_h\| \leq C_{f,q} h^r k^s \tag{3}$$

and there exists a family of data, meaning domains Ω_k, second terms (f_k, q_k) belonging to W, and meshes τ_h characterized by the step size $h = h(k)$, such that the error of the corresponding finite element solution can be estimated as:

$$\frac{e_h}{h^r k^s} \geq Ck^t$$

2.3 On the stabilization of the Helmholtz equation in one dimension with Robin boundary conditions

In order to motivate a minimal requirement on the dependence of h on k, we consider the following model problem:

$$\begin{aligned} -u'' - k^2 u &= f, \quad \text{in } \Omega = (0,1), \\ u(0) &= 0, \\ u'(1) - iku(1) &= 0. \end{aligned} \tag{4}$$

The exact solution is given by $u(x) = \exp(ikx)$. The best approximation in the space of continuous, piecewise linear trial functions satisfies

$$\frac{\|u - u_h\|_{H^s(\Omega)}}{\|u\|_{H^s(\Omega)}} \leq Ch^{2-s} \frac{\|u\|_{H^2(\Omega)}}{\|u\|_{H^s(\Omega)}} \leq C(kh)^{2-s}.$$

Hence, a minimal requirement for the relative error of the best approximation to be small is that kh is small. In this light, the following assumption is very natural

2.4 Control of the Approximation Error

Babuska showed that under certain assumptions. The solution to the finite element solution corresponding to G_{stab} *and* Q_{stab} is pollution-free.

Lemma 1. *Let $f \in H1(0,1)$ and u denote the corresponding solution of (2). Then, the following stability estimate holds*

$$\|u^{(s)}\|_0 \leq Ck^{s-2}\|f\|_{H^1(0,1)}$$

Assume that the right-hand side f of equation (5) belongs to $H^1(0,1)$. Let G^{stab} et Q^{stab} be defined as above, and let U_f be the solution of (3). Then:

$$||(u_f - u)'||_0 \leq ch||f||_1 \quad \text{given that hk} \leq \pi \tag{5}$$

According to Definition 2.1, the stabilized finite element method is free from pollution.

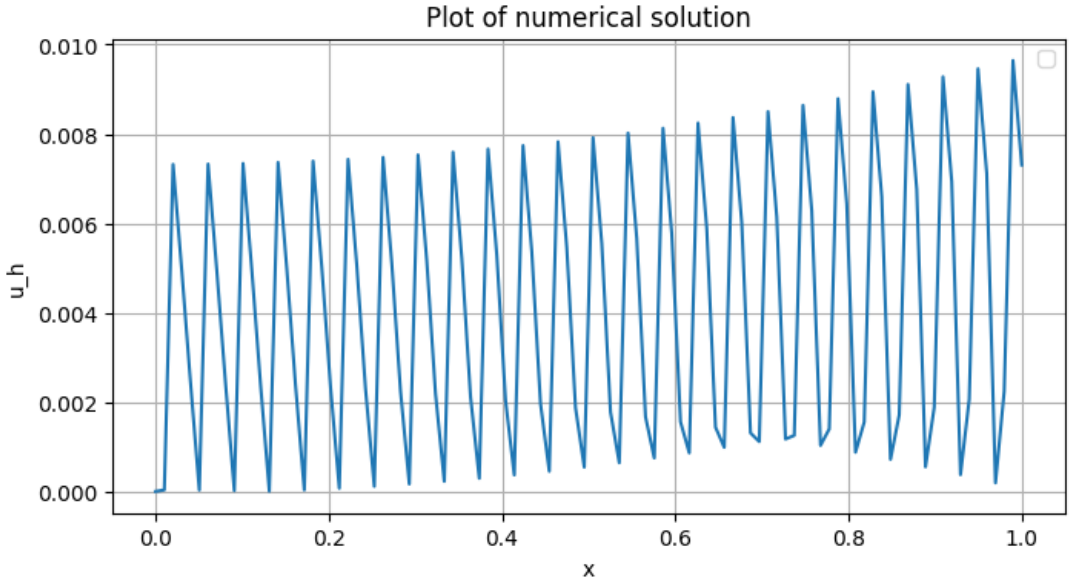
3 Results and discussion

3.1 Numerical examples

In this part, we aim to demonstrate how pollution impacts the Galerkin Finite Element Method (FEM) and the behavior of the stabilized FEM. Our selection for the right-hand side of the equation is $f = 1+x^2$.

We denote by u_h the solution to (1*).

$$\begin{aligned} k &= 50 \\ h &= \frac{\pi}{100 * k} \end{aligned}$$



From the theory of ODE, we know that if $f = 0$, the exact solution takes the form :

$$u_{homogeneous}(x) = Aexp(ikx) + Bexp(-ikx)$$

Because the second term f is polynomial, we have to look for the particular solution of (*) in the form

$$u_p(x) = ax^2 + bx + c$$

Identifying the coefficients, we get :

$$u_{particular}(x) = \frac{-1}{k^2}x^2 + \frac{2}{k^4} - \frac{1}{k^2}$$

Therefore, the exact solution is given by :

$$u(x) = u_{homogeneous} + u_{particular}$$

$$u(x) = Aexp(ikx) + Bexp(-ikx) + \frac{-1}{k^2}x^2 + \frac{2}{k^4} - \frac{1}{k^2}$$

A and B can be found with the help of the boundary conditions.
We then plot the exact solution of (4) :(de Recherche (2024))

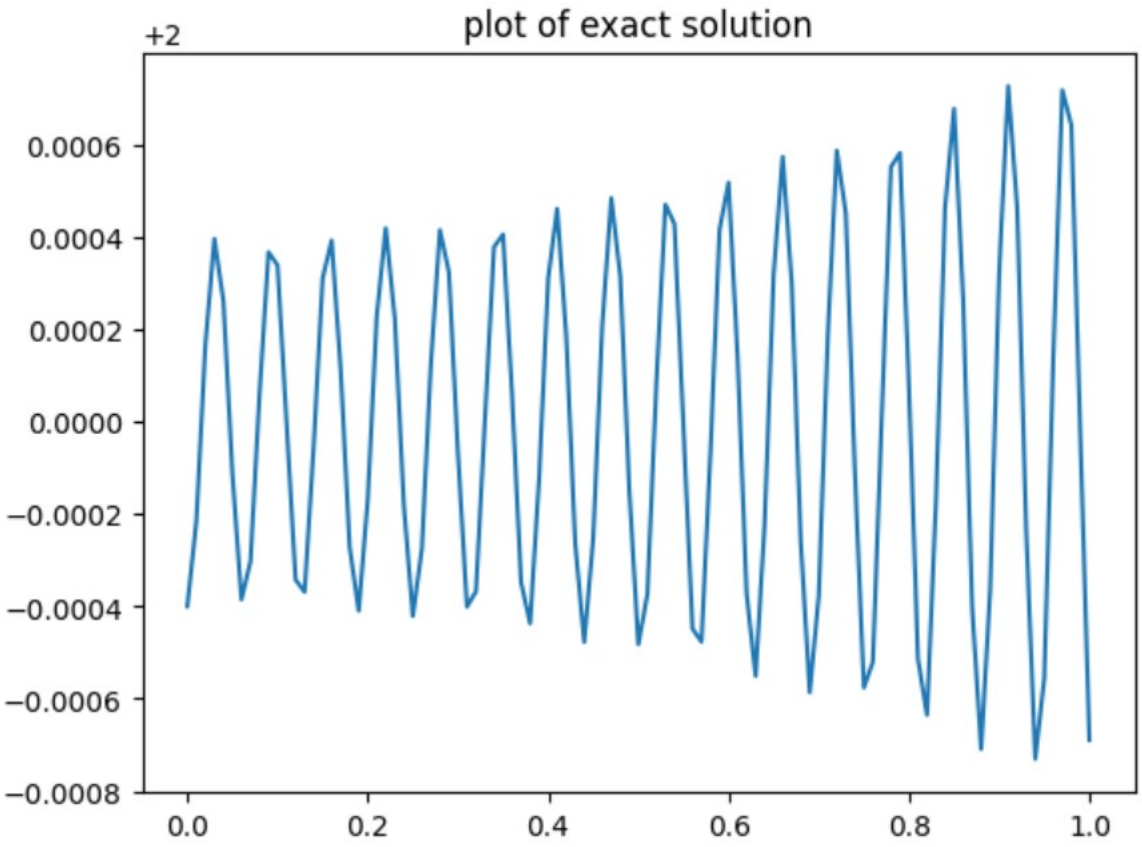
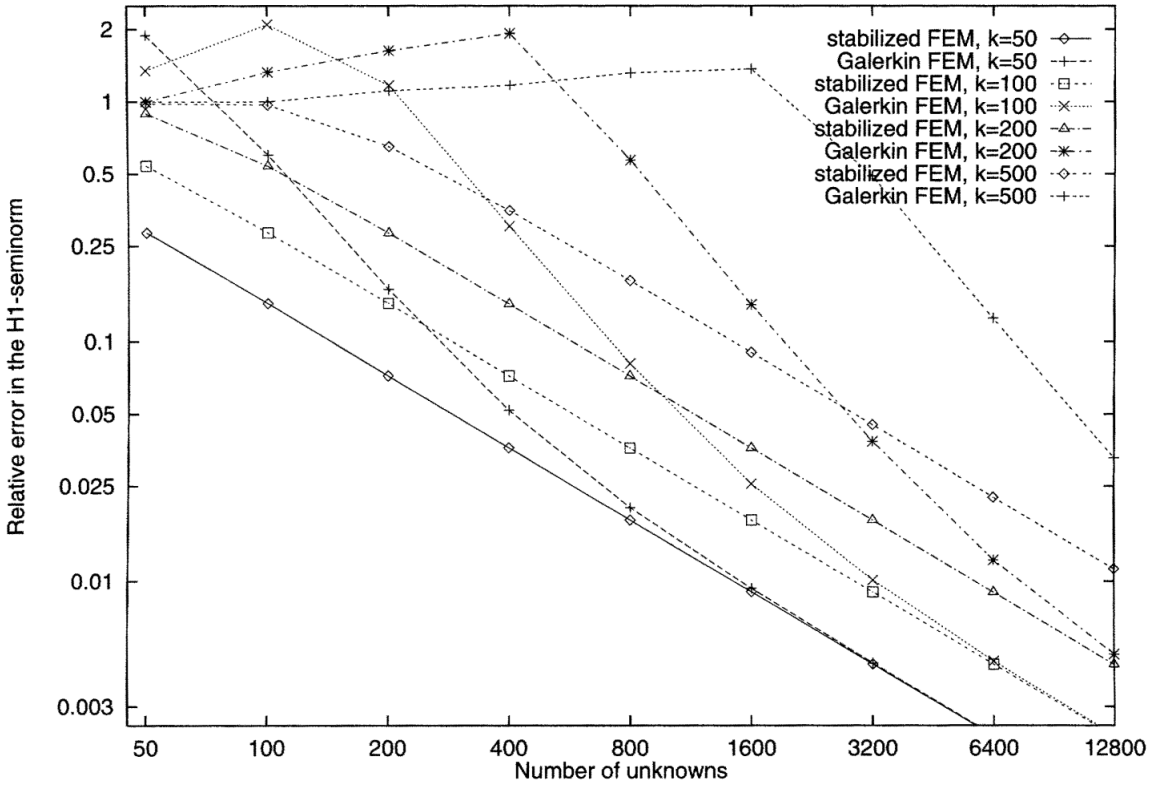


Figure 1: Plot of the exact solution

Furthermore, I.Babuska calculated the H^1 norm of the error for various values of k.
The plot below shows the relative error in the H^1 norm with respect to the number of unkonws.



Figures 4.1 and 4.2 illustrate the impact of pollution in the Galerkin method. As the parameter h decreases, the error curve converges toward the error curve of the optimal approximation. Conversely, for larger values of h, the solution is significantly compromised. The range of h where the Galerkin FEM solution is affected by pollution expands with increasing k. It is noteworthy that even when kh is relatively large, falling within the range [1, 2], the stabilized FEM exhibits the anticipated asymptotic behavior of O(kh). Figure 4.2 demonstrates that the stabilized FEM performs effectively on non-uniform grids, while the pollution effect is more pronounced in the Galerkin FEM.

4 On the stabilization of the Helmholtz equation in two dimensions.

Babuška and Sauter (1997) demonstrated that it's not possible to completely get rid of pollution effects in two dimensions. Using bilinear elements on a regular grid. They looked at increasingly larger areas and used weighted norms to avoid issues at the edges. Despite being a simple model, the conclusions drawn are thought to be applicable to practical situations e.g acoustic waves propagating through the earth's crust.

The central mathematical question revolves around whether, for this straightforward model, it's possible to select coefficients in the equations of a Generalized Finite Element Method (GFEM) in a way that eliminates pollution. If achieving this in the simplified scenario proves impossible, the implication is that it's likely difficult in more general situations as well.

4.1 Setting

Let Ω_n and Q_n be defined by:

$$\Omega_n = (-c_n, c_n)^2, \quad c_n \in N, \quad c_n < c_{n+1}.$$

The homogeneous Helmholtz equation over the domain Ω_n is given by:

$$\begin{cases} \Delta u_n + k^2 u_n = 0 & \text{on} \quad \Omega_n \\ \mathrm{B}_n u_n = r_n & \text{on} \quad \partial\Omega_n \end{cases}$$

We consider GFEM discretizations with piecewise bilinear elements.

Let $h \geq 0$ represent the step size satisfying $h^{-1} \in R$. The grid points are given by $\Theta_n = hZ^2 \cap \Omega_n$. The corresponding Cartesian grid τ_h consists of squares of side length h.

$$\square_\mu = h(\nu_1, \nu_1 + 1) \times h(\nu_2, \nu_2 + 1)$$

$$\tau_h = \square_{\nu\nu \in Z^2} \cap \Omega_n$$

The MFEM is determined by defining a family of regular matrices G_h and functionals Q_h that map the right-hand side of (6.1) to the linear system vector.

We impose additional conditions that any reasonable MFEM method should satisfy.

1. The matrix G_h, depending on k, is sparse in the sense that for each nodal point x inside the domain, the corresponding row of the matrix can be represented by a stencil with nine points.
2. Q_h is local, such that for x inside the domain E_0, the corresponding entry of the right-hand side vector $(Q_h r_n)$ is zero.
3. $G_n^h = \begin{pmatrix} G_2 & G_1 & G_2 \\ G_1 & G_0 & G_1 \\ G_2 & G_1 & G_2 \end{pmatrix}$ for all $x \in \Theta^{int}$.
4. $G_0 = \sum_{m=0}^{\infty} (G_0)_m \alpha^{2m}, G_1 = \sum_{m=0}^{\infty} (G_1)_m \alpha^{2m}, G_2 = \sum_{m=0}^{\infty} (G_2)_m \alpha^{2m}$.
5. We consider the main part of G as an approximation of $a_0(u, v) = \int_\Omega (\nabla u, \nabla v)$.

For each MFEM satisfying the above conditions, there exists a family of domains Ω_n and right-hand sides r_n with $n = n(k, h)$ such that the error of the finite element solution u_h^n compared to the exact solution u_n^{ex} can be estimated as:

$$|u_n^{ex} - u_n^h|_{V-\Omega_n} \geq Ck^{3.5}h^3.$$

The norm $\|\|_-$ is such that $||v||_- = \sup_{w \in H^1(R^2)} \frac{\left|\int_{R^2} \tilde{v}(\sigma)\bar{\omega} d\sigma\right|}{2\pi \|w\|_{H^1(R^2)}}$, where $\tilde{v}$ is the Fourier transform of v.

Thus, for $k \to \infty$ and h chosen such that $k^{3.5}h^3 = 1$, the error of the best approximation tends to zero, while the error of the finite element solution is greater than C. Therefore, the pollution effect is inevitable in two dimensions.

4.2 Numerical Results

In this section we focus on using FreeFEM++, a powerful tool developed by the Laboratoire Jacques Louis Lions at Sorbonne University, Paris. FreeFEM++ provides a user-friendly high-level language.

We propose an implementation of finite element methods for solving the Helmholtz equation in a 2D space.

We consider a rectangular shaped domain Ω.

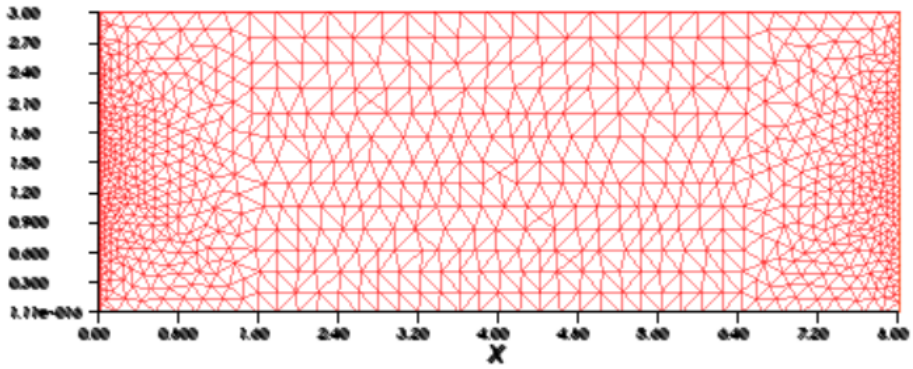
Here, border commands define the boundaries of the domain. Each border corresponds to a side of the L-shaped domain. For example, B1 represents the bottom side, B2 the right side, B3 the top side, and B4 the left side.

```
real aL=8., bL=3.; // geometry
real k2 = 1;
// Define region border
int n=5;
border B1(t=0,aL) { x = t; y=0; label=1; }
border B2(t=0,bL) { x = aL; y=t; label=2; }
border B3(t=0,aL) { x = aL-t; y=bL; label=3; }
border B4(t=0,bL) { x = 0; y=bL-t; label=4; }

mesh Th;

Th = buildmesh(B1(10*n)+B2(8*n)+B3(6*n)+B4(10*n));
plot(Th,wait=true, ps="Lshape.eps");
```

Figure 2: The generated mesh of Ω



The mesh Th; declares a mesh object. The buildmesh command generates the mesh for the L-shaped domain using the specified borders (B1, B2, B3, B4) and boundary discretization parameters$(8 * n, 8 * n, 6 * n, 10 * n)$.

```
real k2=1.; // Wavenumber

fespace Vh(Th,P2); // Define finite element space, test functions fespace
Vh u,v;
// Weak form of Helmholtz operator solve

solve Helmholtz(u,v) = int2d(Th)(dx(u)*dx(v) + dy(u)*dy(v) - k2*u*v) +
on(2,u=1) + on(1,u=0) ;
plot(u, wait=true, ps="Lmode.eps");
```

This formulation sets up the Helmholtz problem in a weak form suitable for finite element analysis, with elements $\in P^2$ and the solve command is used to find the solution u with Dirichlet boundary.

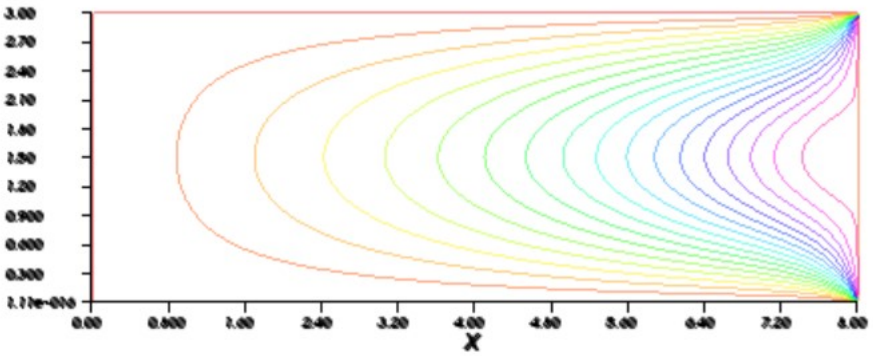


Figure 3: Plot of u

5 The heterogeneous Helmholtz equation

We study the Helmholtz equation defined in the one-dimensional heterogeneous domain $(0, Z)$ with absorbing boundary conditions.

$$\begin{cases} -\frac{\omega^2}{c^2(z)}u(z) - u''(z) = f(z), \quad z \in (0, Z), \\ -u'(0) - \frac{\mathbf{i}\omega}{c(0)}u(0) = 1, \\ u'(Z) - \frac{\mathbf{i}\omega}{c(Z)}u(Z) = 0, \end{cases} \tag{6}$$

Definition 5.1. *We consider parameters c which are piecewise constant and non-increasing. Let* $0 = z_0 < z_1 < \ldots < z_{L-1} < z_L = Z$*, then we will note*

$$c|_{(z_{l-1}, z_l)} = c_l \in R,$$

such that $c_l < c_{l-1}$*. We will also note*

$$\frac{1}{c^2}_l = \frac{1}{c_l^2} - \frac{1}{c_{l-1}^2} > 0.$$

5.1 Stability Results

Theorem 1. *Stability of the Problem under Perturbations of the Velocity c Let* $u \in H^1(0, Z)$ *be a solution of (8). Then,*

$$w^2\|c^{-1}u\|^2 + \|u'\|^2 + 2w^2 \sum_{l=1}^{L-1} \lfloor \frac{1}{c^2} z_l |u(z_l)|^2 \rfloor \leq C_s^2 \|f\|^2.$$

Corollary. *For all* $f \in L^2(0, Z)$ *there exists a unique* $S_{w,c}f \in H^1(0, Z)$*, such that* $B_{w,c}(S_{w,c}f, v) = \int_0^Z f(z)\bar{v(z)}dz$

In the following theorem, we demonstrate that the distance between the operators $S_{\omega,c}$ and S_{ω,c_ϵ} can be controlled by the distance between c and c_ϵ in a suitable norm.

Theorem 2. *For all* $f \in L^2(0, Z)$*:*

$$\|S_{\omega,c}f - S_{\omega,c_\epsilon}f\|_{w,c} \leq C_{s,q} w^2 \|c^{-2} - c_\epsilon^{-2}\|_1 \|f\|.$$

Where $C_{s,q}$ *is a constant.*

5.2 Finite Element Discretization

Theorem 3. *Assuming that* $\mu_{w,c}^{h,p} \leq \rho$ *Then for all* $f \in L^2(0, Z)$*, there exists a unique element* $S_{w,c}^{h,p} f \in V^{h,p}$ *such that :*

$$B_{w,c}(S^{h,p}_{w,c}f, v_h) = \int_0^Z f(z)\bar{v_h}(z)dz \qquad \text{for all } v_h \in S_{w,c}.$$

(7)

We introduce a discretization step $h = 1/n_h$ with $n_h \in N$, and the associated decomposition $t_j = jh$, for $j \in \{0, \ldots, n_h\}$.

Next, we define the discretization space as follows:

$$V^{h,p} = \{v \in H^1(0, Z) \mid v_{(t_{j-1},t_j)} \in \mathcal{P}, 0 \leq j \leq n_h\}$$

We do not assume that the mesh adapts to the jumps in the wave velocity c. Thus, it is possible that c has jumps inside a mesh cell. While assuming mesh adaptation to c jumps would greatly simplify the analysis, it is crucial to consider the general case. Indeed, in highly heterogeneous media, mesh adaptation to c jumps could result in significant computational burden. On the other hand, our analysis naturally covers coarse meshes, where the velocity parameter can change arbitrarily within the cell.

5.3 Multiscale Medium Approximation Method

The search for the solution to the variational problem equivalent to (8) implies that we are able to compute the coefficients of the linear system. Among these coefficients are the integrals

$$\int_0^Z \frac{1}{c^2}\phi_h(z)\psi_h(z)dz \text{ for all } \phi_h, \psi_h \in V^{h,p}$$

In one dimension , the integral can be computed analytically, but not in 2D. Even with an analytical formula (e.g., if the interfaces are polygons), the calculation can be costly as the quadrature varies between cells. We propose an alternative approach: a cost-effective numerical approximation with an alternative parameter c_ϵ. This is the **Multiscale Medium Approximation Method.** *MMAm*

Let $\phi_h \in V_h$ and a mesh cell (t_{j-1}, t_j). Either $\phi|_{(t_{j-1},t_j)}(z) = \hat{l}_h(\frac{z-t_{j-1}}{t_j-t_{j-1}})$ or $\hat{l}_\mu{}_{\mu=0}^{p+1}$ is the Lagrangian basis of (0,1).

It is evident that calculating () for each combination of basis functions amounts to computing the integrals.

$$\int_{t_{j-1}}^{t_j} \frac{1}{c^2(z)}\hat{l}_\mu\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)\hat{l}_\lambda\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)dz \tag{8}$$

In finite element analysis, to accelerate the computations of (37), we precompute the quantities with the Lagrangian basis of the reference cell. In the standard FEM method, these reference values $\hat{M}_{\mu\lambda} = \int_0^1 \hat{l}_\mu(\hat{z})\hat{l}_\lambda(\hat{z})$ are calculated, and since c is assumed to be constant within each cell,

$$\int_{t_{j-1}}^{t_j} \frac{1}{c^2(z)}\hat{l}_\mu\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)\hat{l}_\lambda\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)dz = \frac{h}{c_j}\hat{M}_{\mu\lambda}$$

The technique used in MMAm is similar to the standard finite element method but allows c to take m different values within each cell.

$M^i_{\mu\lambda} = \int_{\tilde{t}_i}^{\tilde{t}_{i-1}} \hat{l}_\mu(\hat{z})\hat{l}_\lambda(\hat{z})d\hat{z}$ with $\tilde{t}_i = \frac{i}{m}$ are calculated for $1 \leq i \leq m$ and $1 \leq \lambda, \mu \leq p+1$.

So that :

$$\begin{aligned}\int_{t_j}^{t_{j-1}} \frac{1}{c^2(z)}\hat{l}_\mu\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)\hat{l}_\lambda\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)dz &= \sum_{i=0}^{m}\int_{t_j^{i-1}}^{t_j^i} \frac{1}{c^2(z)}\hat{l}_\mu\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)\hat{l}_\lambda\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)dz \\ &= \sum_0^m \frac{1}{c^2_{j,i}}\int_{t_j^{i-1}}^{t_j^i} \hat{l}_\mu\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)\hat{l}_\lambda\left(\frac{z-t_{j-1}}{t_j-t_{j-1}}\right)dz\end{aligned}$$

$$= (\mathrm{t}_j - t_{j-1}) \sum_{i=1}^{m} \frac{1}{c_{j,i}^2} \int_{\hat{t}_j}^{\hat{t_{j-1}}} \hat{l}_\mu(\hat{z}) \hat{l}_\lambda(\hat{z}) d\hat{z}$$
$$= \mathrm{h} \sum_{i=1}^{m} \frac{1}{c_{j,i}^2} M_{\mu\lambda}^i$$

The computation of the linear system is m times more costly in MMAm than in the standard finite element method. The construction of the linear system in the standard method is in $O(np^{2d})$, whereas for MMAm with m sub-quadrature cells, the complexity is in $O(mnp^{2d})$. However, this step can be easily parallelized as the calculations are local, with no need for communication between cells.

5.4 Non-fitting vs fitting mesh

In the realm of finite element methods (FEM) addressing interface problems, two primary classes emerge: fitted-mesh FEM and unfitted-mesh FEM.Chen, Hou, and Zhang (2020) The fitted-mesh approach, demands that the solution mesh align precisely with the interface. Deviation from this alignment threatens the convergence of the numerical method. However, this body-fitting constraint imposes limitations, particularly in scenarios featuring a dynamic or moving interface in our case the c is considered to be piecewise constant. In such cases, the solution mesh must undergo regeneration at each time level, rendering the fitted-mesh method less versatile.Chen and Zhang (2021)

In contrast, unfitted-mesh methods offer a more flexible alternative. These methods tend to alleviate, and in some cases, entirely eliminate the restrictions imposed on the mesh. By doing so, they provide a solution that is not hindered by the need for continuous realignment with the interface. This characteristic makes unfitted-mesh methods particularly well-suited for problems involving dynamic interfaces or scenarios where the geometry undergoes significant changes over time. The inherent adaptability of unfitted-mesh methods enhances their applicability across a broader range of interface problems, making them a valuable choice in diverse simulation and modeling contexts.

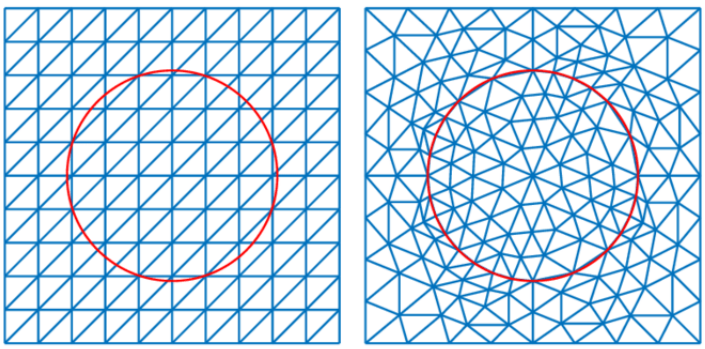


Figure 4: Non-body fitting (left) and body-fitting (right) meshes

6 Conclusion

In conclusion, in two and more space dimensions, it is impossible to eliminate pollution effect. Nevertheless, our study of the numerical solution of the Helmholtz equation shed light on crucial aspects related to the use of methods such as MMAm in diverse environments.

The results highlight the effectiveness of high-order polynomials and higher-order discretizations but also underscore persistent challenges in highly heterogeneous environments, especially as the wavenumber k tends to infinity.

The importance of choosing methods tailored to the characteristics of the problem is evident from our analysis. While significant progress has been made, uncertainties remain, particularly in the pre-asymptotic error estimation and the guarantee of a constant error.

The key findings from our exploration are as follows:

1. Advantages of High-Order Polynomials: MMAm applied to the one-dimensional Helmholtz problem demonstrates the benefits of high-order polynomials over low-order ones, leading to better convergence and increased accuracy of the numerical solution.
2. Pre-Asymptotic Error Estimation: Despite detailed analysis, pre-asymptotic error estimation in heterogeneous environments remains an open question, requiring further research for a thorough understanding.
3. Superiority of Higher-Order Discretizations: Numerical experiments confirm the superiority of higher-order discretizations on non-adaptive meshes, especially in environments with varied velocity contrasts.
4. Challenges Related to Pollution Effect: The pollution effect in the numerical resolution of the Helmholtz equation, particularly as k tends to infinity, represents a major challenge requiring robust numerical methods.

The results emphasize the importance of choosing appropriate numerical methods, especially in heterogeneous environments. The effectiveness of high-order polynomials and higher-order discretizations is highlighted, but caution is necessary in highly heterogeneous contexts.

7 Future Directions and Perspectives

The outlook of this study suggests several perspectives for future research:

- Further investigation into pre-asymptotic error estimation in heterogeneous environments.
- Exploration of guarantees for constant error in highly heterogeneous environments.
- Extension of MMAm to three-dimensional problems and incorporation of adaptive strategies.

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