Convergence of a sequence

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Question 1:

Let's prove this property by induction on n.

Initialization: $u_2 = 3$, so $u_2u_0 - u_1^2 = -1$. The property is verified for $n = 1$.

Induction: Assume the property is verified at rank n and show it at rank $n+1$. By the induction hypothesis:

$$u_{n+2}u_n - u_{n+1}^2 = u_n^2 - u_{n+1}u_{n-1} = -(-1)^n = (-1)^{n+1}$$

This concludes the induction.

Question 2:

$(Z/dZ)^2$ has a cardinality of d^2, so by the Pigeonhole Principle, there exist $0 \leq i < j \leq d^2$ such that $(u_i, u_{i+1}) \equiv (u_j, u_{j+1})\,[d]$.

As $u_n \equiv 3u_{n-1} + u_{n-2}\,[d]$ and $u_{n-2} \equiv u_n - 3u_{n-1}\,[d]$ for $n \geq 2$, by double rising and double descending induction, it is deduced that $u_{i+n} \equiv u_{j+n}\,[d]$ for all $n \geq -\min(i, j)$, implying that $(u_n\,[d])$ is $j - i$ periodic, hence periodic.

Question 3:

Assume there exists k such that $u_k \equiv u_{k+1} \equiv 0\,[d]$. As $u_{n-2} \equiv u_n - 3u_{n-1}\,[d]$ for $n \geq 2$, by double descending induction, it is deduced that $u_i \equiv u_{i+1} \equiv 0\,[d]$ for $i \geq 0$, and in particular, $u_1 \equiv 0\,[d]$, which is absurd.

Therefore, $(u_i, u_{i+1})\,[d]$ belongs to $(Z/dZ)^2 \backslash \{(0,0)\}$ with a cardinality of d^2-1. By the Pigeonhole Principle, there exist $0 \leq i < j \leq d^2 - 1$ such that $(u_i, u_{i+1}) \equiv (u_j, u_{j+1})\,[d]$. $(u_n\,[d])$ is thus $j - i$ periodic, so the smallest period T satisfies $T \leq d^2 - 1$.

Question 4:

If $(u_n\,[d])$ is T-periodic, then for $n \geq 2$:

$$u_{n+1}u_{n-1} - u_n^2 \equiv u_{n+1+T}u_{n-1+T} - u_{n+T}^2\,[d]$$

Using Question 1, it is deduced that $(-1)^n \equiv (-1)^{n+T}\,[d]$, and thus $(-1)^T \equiv 1\,[d]$, which implies T is even for $d > 2$.