Probability of invertibility

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Question 1:

Let $\chi_M = \det \begin{pmatrix} X_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{n-1} \end{pmatrix}$.

Expanding with respect to the first column:

$$\chi_M = \det \begin{pmatrix} X & 0 & \cdots & 0 \\ -X_{n-1} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & X_{n-1} \\ 0 & \cdots & 0 & -X \end{pmatrix} + (-1)^n \det \begin{pmatrix} -X_{n-1} & \cdots & 0 \\ X & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{n-1} \end{pmatrix}$$
$$= X \cdot X^{n-1} + (-1)^n \cdot (-1)^{n-1}$$
$$= X^n - 1.$$

Note $\xi = e^{\frac{2i\pi}{n}}$. χ_M has n simple roots, which are ξ_k for $k \in [1, n]$. Since the space is of dimension n, the eigenspaces E_{ξ_k} are of dimension 1.

Let V_k be an eigenvector of M associated with ξ_k. $MV_k = \xi_k V_k$. Therefore:

$$\begin{cases} x_1 = \xi_k x_n \\ x_j = \xi_k x_{j-1}, \quad j \in [2, n] \end{cases}$$

Thus:

$$E_{\xi_k} = \mathrm{Vect} \left\{ \begin{pmatrix} 1 \\ \xi_k \\ \vdots \\ \xi_k^{n-1} \end{pmatrix} \right\}$$

Question 2:

$$P(A \notin GL_n(\mathbb{C})) = P(\det(A) = 0).$$

Note that $A = \sum_{j=0}^{n-1} X_j M^j = P(M)$ with $P(X) = \sum_{j=0}^{n-1} X_j X^j$.

Let $\forall k \in [1, n]$, $V_k = \begin{pmatrix} 1 & \xi_k & \dots & \xi_k^{n-1} \end{pmatrix}^T$.

$(V_1, \ldots, V_n)$ forms a basis of eigenvectors of M. $(V_1, \ldots, V_n)$ also forms a basis of eigenvectors of $A = P(M)$ associated with $P(\xi_k)$.

$$\det(A) = \prod_{k=1}^{n} P(\xi_k).$$

We deduce:

$$P(A \notin GL_n(\mathbb{C})) = P\left(\bigcup_{k=1}^{n} P(\xi_k) = 0\right).$$

For $j \in [1, n-1]$, let $I_j = \{T \in \mathbb{Q}[X] \mid T(\xi_j) = 0\}$. I_j is an ideal of the ring $\mathbb{Q}[X]$, so there exists a polynomial $\Pi_j \in \mathbb{Q}[X]$ such that $I_j = \Pi_j \mathbb{Q}[X]$.

$$X^{n-1} + \ldots + X + 1 \in I_j$$

As $\xi_j^{-1} \neq 0$, we have $1 + \xi_j^{-1} + \ldots + \xi_j^{-(n-1)} \in I_j$. Therefore, there exists $T \in \mathbb{Q}[X]$ such that $1 + \xi_j^{-1} + \ldots + \xi_j^{-(n-1)} = \Pi_j T$. Now, $1 + \xi_j + \ldots + \xi_j^{n-1}$